

# The Transcendence Degree over a Ring

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## Abstract

For a finitely generated algebra over a field, the transcendence degree is known to be equal to the Krull dimension. The aim of this paper is to generalize this result to algebras over rings. A new definition of the transcendence degree of an algebra  $A$  over a ring  $R$  is given by calling elements of  $A$  algebraically dependent if they satisfy an algebraic equation over  $R$  whose trailing coefficient, with respect to some monomial ordering, is 1. The main result is that for a finitely generated algebra over a Noetherian Jacobson ring, the transcendence degree is equal to the Krull dimension.

## Introduction

The equality of Krull dimension and transcendence degree for a finitely generated algebra over a field is one of the fundamental results in commutative algebra. Various extensions of this result have appeared in the literature. Onoda and Yoshida [11] generalized the result to subalgebras of a finitely generated algebra over a field. Tanimoto [13] showed that for a finitely generated domain  $A$  over a field and a prime ideal  $P \in \text{Spec}(A)$ , the local ring  $A_P$  has a subfield  $L$  such that the transcendence degree of  $A_P$  over  $L$  equals  $\dim(A_P)$ . Some authors, among them Giral [5] and Hamann [6], proposed several notions of a transcendence degree of an algebra over a ring and studied their behavior. In the introduction, Hamann wrote that her paper might better be titled: “Why there is no notion of transcendence degree over arbitrary commutative rings.”

This paper aims to take up the challenge posed by this comment. We give a new definition of the transcendence degree of an algebra  $A$  over a ring  $R$  by calling elements  $a_1, \dots, a_n \in A$  algebraically dependent if they satisfy an equation  $f(a_1, \dots, a_n) = 0$ , where  $f$  a polynomial with coefficients in  $R$  such that the trailing coefficient of  $f$ , with respect to some monomial ordering, is 1 (see Definition 1.1). If  $R$  is a field, this definition coincides with the usual one. But in other cases, the new transcendence degree behaves in unexpected ways. For example, the transcendence degree of a ring over itself is “usually” not zero. As the main result, we prove that if  $A$  is finitely generated and  $R$  is a Noetherian Jacobson ring, then the Krull dimension of  $A$  is equal to the transcendence degree of  $A$  over  $R$ . In fact, this result extends to the case that  $A$  is contained in a finitely generated  $R$ -algebra, and if  $A = R$ , the hypothesis that  $R$  is a Jacobson ring can be dropped (see Theorem 1.4). In the case  $A = R$  the result was already proved for the lexicographic monomial ordering by Coquand and Lombardi [3] (see Theorem 1.3).

The paper is organized as follows. The first section contains the new definition of the transcendence degree, some examples, and the statement of the main result (Theorem 1.4). This is proved in the second section. We also show that the validity of Theorem 1.4 characterizes Jacobson rings (see Remark 2.7). The last section is devoted to some applications and to the question whether the transcendence degree depends of the choice of a monomial ordering. We conjecture that it does not (Conjecture 3.8), and prove some special cases (see Theorem 3.7).

This work was inspired by reading the above-mentioned paper of Coquand and Lombardi [3], who characterized the Krull dimension by certain types of identities. Interpreting this in terms

of the lexicographic monomial ordering led to the new definition of the transcendence degree and prompted the questions to what extent this depends on the choice of the monomial ordering, and whether one can also prove a “relative” version involving the transcendence degree over a subring. In his bachelor thesis [1], Christoph Bärligea studied (among other things) the first question and found no example where the transcendence degree depends on the monomial ordering. I wish to thank Peter Heinig for bringing Coquand and Lombardi’s article to my attention.

## 1 A new definition of the transcendence degree

All rings in this paper are assumed to be commutative with an identity element 1. If  $R$  is a ring, an  $R$ -algebra is a ring  $A$  together with a ring homomorphism  $R \rightarrow A$ . We call an  $R$ -algebra **subfinite** if it is a subalgebra of a finitely generated  $R$ -algebra. By  $\dim(R)$  we will always mean the Krull dimension of  $R$ . We follow the convention that the zero ring  $R = \{0\}$  has Krull dimension  $-1$ . It will be convenient to work with the polynomial ring  $R[x_1, x_2, \dots]$  with infinitely many indeterminates over a ring  $R$ , and to understand a **monomial ordering** as a total ordering “ $\preceq$ ” on the set of monomials of  $R[x_1, x_2, \dots]$  such that the conditions  $1 \preceq s$  and  $st_1 \preceq st_2$  hold for all monomials  $s, t_1$ , and  $t_2$  with  $t_1 \preceq t_2$ . Clearly any monomial ordering on a polynomial ring  $R[x_1, \dots, x_n]$  with finitely many indeterminates can be extended to a monomial ordering in the above sense.

The following notions of algebraic dependence and transcendence degree over a ring generalize the corresponding notions over a field.

**Definition 1.1.** *Let  $R$  be a ring.*

- (a) *Let “ $\preceq$ ” be a monomial ordering. A nonzero polynomial  $f \in R[x_1, x_2, \dots]$  is called **submonic** with respect to “ $\preceq$ ” if its trailing coefficient (i.e., the coefficient of the least monomial having nonzero coefficient) is 1.*

*A polynomial is called **submonic** if there exists a monomial ordering with respect to which it is submonic.*

- (b) *Let  $A$  be an  $R$ -algebra. Elements  $a_1, \dots, a_n \in A$  are called **algebraically dependent** over  $R$  if there exists a submonic polynomial  $f \in R[x_1, \dots, x_n]$  such that  $f(a_1, \dots, a_n) = 0$ . (Of course the homomorphism  $R \rightarrow A$  is applied to the coefficients of  $f$  before evaluating at  $a_1, \dots, a_n$ .) Otherwise,  $a_1, \dots, a_n$  are called **algebraically independent** over  $R$ .*

*We can also restrict  $f$  to be submonic with respect to a specified monomial ordering “ $\preceq$ ”, in which case we speak of algebraic dependence or independence with respect to “ $\preceq$ ”.*

- (c) *For an  $R$ -algebra  $A$ , the **transcendence degree** of  $A$  over  $R$  is defined as*

$$\text{trdeg}(A : R) := \sup \{ n \in \mathbb{N} \mid \text{there exist } a_1, \dots, a_n \in A \text{ that are algebraically independent over } R \}.$$

*If every  $a \in A$  is algebraically dependent over  $R$ , we set  $\text{trdeg}(A : R) := 0$  in the case  $A \neq \{0\}$  and  $\text{trdeg}(A : R) := -1$  in the case  $A = \{0\}$ . We write  $\text{trdeg}(R) := \text{trdeg}(R : R)$  for the transcendence degree of  $R$  over itself.*

*If “ $\preceq$ ” is a monomial ordering, we define  $\text{trdeg}_{\preceq}(A : R)$  by requiring algebraic independence with respect to “ $\preceq$ ”.*

**Example 1.2.** (1) If  $R$  is an integral domain, then the elements of  $R$  that are algebraically dependent over  $R$  are 0 and the units of  $R$ .

- (2) If  $R$  is a nonzero finite ring, then  $\text{trdeg}(R) = 0$  since for each  $a \in R$  there exist nonnegative integers  $m < n$  such that  $a^m = a^n$ .

- (3) The following example shows that the notion of algebraic dependence with respect to a monomial ordering depends on the chosen monomial ordering. Let  $R = K[t_1, t_2]$  be a polynomial ring in two indeterminates and let  $a = t_1$  and  $b = t_1 t_2$ . The relation  $b - t_2 a = 0$  tells us that  $a, b$  are algebraically dependent over  $R$  with respect to the lexicographic monomial ordering with  $x_1 > x_2$ . On the other hand, algebraic dependence over  $R$  with respect to the lexicographic ordering with  $x_2 > x_1$  would mean that there exist  $i, j \in \mathbb{N}_0$  such that  $a^i b^j = t_1^{i+j} t_2^j$  lies in the  $R$ -ideal

$$(b^{j+1}, a^{i+1} b^j)_R = \left( (t_1 t_2)^{j+1}, t_1^{i+j+1} t_2^j \right)_R,$$

which is not the case.

- (4) We consider  $R = \mathbb{Z}$  and claim that  $\text{trdeg}(\mathbb{Z}) = 1$ . Since  $\mathbb{Z}$  has nonzero elements which are not units, we have  $\text{trdeg}(\mathbb{Z}) \geq 1$ . We need to show that all pairs of integers  $a, b \in \mathbb{Z}$  are algebraically dependent over  $\mathbb{Z}$ . We may assume  $a$  and  $b$  to be nonzero and write

$$a = \pm \prod_{i=1}^r p_i^{d_i} \quad \text{and} \quad b = \pm \prod_{i=1}^r p_i^{e_i},$$

where the  $p_i$  are pairwise distinct prime numbers and  $d_i, e_i \in \mathbb{N}_0$ . Choose  $n \in \mathbb{N}_0$  such that  $n \geq d_i/e_i$  for all  $i$  with  $e_i > 0$ . Then

$$\gcd(a, b^{n+1}) = \prod_{i=1}^r p_i^{\min\{d_i, (n+1)e_i\}} \quad \text{divides} \quad \prod_{i=1}^r p_i^{n e_i} = b^n,$$

so there exist  $c, d \in \mathbb{Z}$  such that  $b^n = ca + db^{n+1}$ . Since  $f = x_2^n - cx_1 - dx_2^{n+1}$  is submonic (with respect to the lexicographic ordering with  $x_1 > x_2$ ), this shows that  $a, b$  are algebraically dependent.

Clearly this argument carries over to any principal ideal domain that is not a field. It is remarkable that although the transcendence degree is an algebraic invariant, the above calculation has a distinctly arithmetic flavor.  $\triangleleft$

It becomes clear from Example 1.2(1) that sums of algebraic elements need not be algebraic, and from (4) that the transcendence degree does not behave additively for towers of ring extensions.

Coquand and Lombardi [3] proved that for a ring  $R$  and an integer  $n \in \mathbb{N}$ , the inequality  $\dim(R) < n$  holds if and only if for all  $a_1, \dots, a_n \in R$  there exist  $m_1, \dots, m_n \in \mathbb{N}_0$  such that

$$\prod_{i=1}^n a_i^{m_i} \in \left( a_j \cdot \prod_{i=1}^j a_i^{m_i} \mid j = 1, \dots, n \right)_R \quad (1.1)$$

(also see Kemper [8, Exercise 6.8]). Using Definition 1.1 and writing  $\text{lex}$  for the lexicographic ordering with  $x_i > x_{i+1}$  for all  $i$ , we can reformulate this result as follows.

**Theorem 1.3** (Coquand and Lombardi [3]). *If  $R$  is a ring, then*

$$\text{trdeg}_{\text{lex}}(R) = \dim(R).$$

The following is the main result of this paper.

**Theorem 1.4.** (a) *If  $R$  is a Noetherian ring, then*

$$\text{trdeg}(R) = \dim(R).$$

(b) *If  $R$  is a Noetherian Jacobson ring and  $A$  is a subfinite  $R$ -algebra, then*

$$\text{trdeg}_{\text{lex}}(A : R) = \dim(A).$$

(c) If  $R$  and  $A$  are as in (b) and  $A$  is Noetherian, then

$$\text{trdeg}(A : R) = \dim(A).$$

Parts (b) and (c) generalize the classical result that the Krull dimension of a finitely generated algebra over a field is equal to its transcendence degree.

## 2 Proof of the main result

The proof of Theorem 1.4 is subdivided into various steps, which contain results that are themselves of some interest.

Recall that in a Noetherian ring  $R$  with  $\dim(R) \geq n$  there exist elements  $a_1, \dots, a_n$  and a prime ideal  $P$  of height  $n$  that lies minimally over  $(a_1, \dots, a_n)_R$  (see [8, Theorem 7.8]). Therefore the following theorem implies the inequality

$$\dim(R) \leq \text{trdeg}(R). \quad (2.1)$$

**Theorem 2.1.** *Let  $R$  be a Noetherian ring. If  $a_1, \dots, a_n \in R$  are elements such that  $R$  has a prime ideal of height  $n$  lying minimally over  $(a_1, \dots, a_n)_R$ , then the  $a_i$  are algebraically independent over  $R$ .*

*Proof.* Let  $P \in \text{Spec}(R)$  be a prime ideal of height  $n$  lying minimally over  $(a_1, \dots, a_n)_R$ . Clearly it suffices to show that the images of the  $a_i$  in the localization  $R_P$  are algebraically independent. Substituting  $R$  by  $R_P$ , we may therefore assume that  $R$  is a local ring and  $a_1, \dots, a_n$  form a system of parameters. With  $\mathfrak{q} := (a_1, \dots, a_n)_R$ , this implies that for all  $j \in \mathbb{N}_0$  the module  $R/\mathfrak{q}^j$  has finite length, and for sufficiently large  $j$  this length is given by a polynomial of degree  $n$  in  $j$  (see Matsumura [9, Theorem 17]).

By way of contradiction, assume that  $a_1, \dots, a_n$  are algebraically dependent. This means that there exists a monomial  $t := \prod_{i=1}^n x_i^{d_i}$  and further monomials  $t_k := \prod_{i=1}^n x_i^{e_{k,i}}$  that are greater than  $t$  with respect to some monomial ordering such that

$$\prod_{i=1}^n a_i^{d_i} = \sum_{k=1}^m r_k \prod_{i=1}^n a_i^{e_{k,i}}, \quad (2.2)$$

where the  $r_k$  are elements of  $R$ . By [8, Exercise 9.2(b)], there exist positive integers  $w_1, \dots, w_n$  such that  $\sum_{i=1}^n w_i d_i < \sum_{i=1}^n w_i e_{k,i}$  holds for  $k \in \{1, \dots, m\}$ . (The exercise uses the so-called convex cone of the monomial ordering, and a solution is provided in the book.) Writing  $(\underline{d}) = (d_1, \dots, d_n)$  and  $w(\underline{d}) := \sum_{i=1}^n w_i d_i$ , we can express the inequalities as

$$w(\underline{d}) < w(\underline{e}_k) \quad (k \in \{1, \dots, m\}). \quad (2.3)$$

For  $j \in \mathbb{N}_0$  we define the ideal

$$I_j := \left( \prod_{i=1}^n a_i^{e_i} \mid e_1, \dots, e_n \in \mathbb{N}_0, w(\underline{e}) \geq j \right)_R \subseteq R.$$

With  $\overline{w} := \max\{w_1, \dots, w_n\}$  we have  $I_{\overline{w}j} \subseteq \mathfrak{q}^j$  for all  $j$ , so for  $j$  sufficiently large, the length of  $R/I_{\overline{w}j}$  is bounded below by a polynomial of degree  $n$  in  $j$ .

Clearly  $I_{j+1} \subseteq I_j$ . We write

$$m_j := \left| \left\{ (e_1, \dots, e_n) \in \mathbb{N}_0^n \mid w(\underline{e}) = j \right\} \right| \quad (j \in \mathbb{Z})$$

and claim that  $I_j/I_{j+1}$  is generated as an  $R$ -module by  $m_j - m_{j-w(\underline{d})}$  elements. In fact,  $I_j/I_{j+1}$  is clearly generated by all  $\prod_{i=1}^n a_i^{e_i}$  with  $w(\underline{e}) = j$ . But if  $(\underline{e}) \in \mathbb{N}_0^n$  has the form  $(\underline{e}) = (\underline{d}) + (\underline{e}')$  with  $w(\underline{e}') = j - w(\underline{d})$ , then

$$\prod_{i=1}^n a_i^{e_i} = \prod_{i=1}^n a_i^{d_i + e'_i} \stackrel{(2.2)}{=} \sum_{k=1}^m r_k \prod_{i=1}^n a_i^{e_{k,i} + e'_i} \in I_{j+1},$$

since

$$w(\underline{e}_k + \underline{e}') = w(\underline{e}_k) + w(\underline{e}') \underset{(2.3)}{>} w(\underline{d}) + w(\underline{e}') = j.$$

So we can exclude such a product  $\prod_{i=1}^n a_i^{e_i}$  from our set of generators and are left with  $m_j - m_{j-w(\underline{d})}$  generators, as claimed. The definition of  $I_j$  implies that  $\mathfrak{q}I_j \subseteq I_{j+1}$ , so  $I_j/I_{j+1}$  is an  $R/\mathfrak{q}$ -module. With  $l_0 := \text{length}(R/\mathfrak{q})$ , we obtain

$$\text{length}(R/I_j) = \sum_{i=0}^{j-1} \text{length}(I_i/I_{i+1}) \leq l_0 \sum_{i=0}^{j-1} (m_i - m_{i-w(\underline{d})}) = l_0 \sum_{i=j-w(\underline{d})}^{j-1} m_i.$$

In the ring  $\mathbb{Z}[[t]]$  of formal power series over  $\mathbb{Z}$ , we have

$$\begin{aligned} \sum_{j=0}^{\infty} \left( l_0 \sum_{i=j-w(\underline{d})}^{j-1} m_i \right) t^j &= l_0 \sum_{i=0}^{\infty} \sum_{j=i+1}^{i+w(\underline{d})} m_i t^j = l_0 \left( \sum_{i=0}^{\infty} m_i t^i \right) \left( \sum_{j=1}^{w(\underline{d})} t^j \right) \\ &= \frac{l_0 (t + t^2 + \cdots + t^{w(\underline{d})})}{(1 - t^{w_1}) \cdots (1 - t^{w_n})} = \frac{g(t)}{(1 - t^{w_0})^n} = g(t) \cdot \sum_{j=0}^{\infty} \binom{j+n-1}{n-1} t^{w_0 j}, \end{aligned}$$

where  $w_0 := \text{lcm}(w_1, \dots, w_n)$  and  $g(t) \in \mathbb{Z}[t]$ . It follows that  $\text{length}(R/I_j)$  is bounded above by a polynomial of degree at most  $n-1$  in  $j$ , contradicting the fact that  $\text{length}(R/I_{w_j})$  is bounded below by a polynomial of degree  $n$  for large  $j$ . This contradiction finishes the proof.  $\square$

**Remark.** The converse statement of Theorem 2.1 may fail: For example, the only prime ideal lying minimally over the class of  $x$  in  $R := K[x, y]/(x \cdot y)$  (with  $K$  a field and  $x$  and  $y$  indeterminates) has height 0, but the class of  $x$  is nevertheless algebraically independent over  $R$ .  $\triangleleft$

*Proof of Theorem 1.4(a).* For any monomial ordering “ $\preceq$ ”, it follows directly from Definition 1.1 that  $\text{trdeg}(R) \leq \text{trdeg}_{\preceq}(R)$ . Applying this to the lexicographic ordering and using (2.1) and Theorem 1.3 yields Theorem 1.4(a).  $\square$

Let  $A$  be an algebra over a ring  $R$  and let “ $\preceq$ ” be a monomial ordering. Then the inequalities

$$\text{trdeg}(A) \leq \text{trdeg}(A : R) \leq \text{trdeg}_{\preceq}(A : R) \quad (2.4)$$

follow directly from Definition 1.1. The next goal is to prove  $\text{trdeg}_{\text{lex}}(A : R) \leq \dim(A)$  in the case that  $R$  is a Noetherian Jacobson ring and  $A$  is finitely generated over  $R$ . To achieve this goal, we need four lemmas. The proof of part (a) of the following lemma was shown to me by Viet-Trung Ngo.

**Lemma 2.2.** *Let  $R$  be a Noetherian ring and  $P \in \text{Spec}(R)$ .*

- (a) *If  $Q \in \text{Spec}(R)$  such that  $P \subseteq Q$  and  $Q/P \in \text{Spec}(R/P)$  has height at least 2, then there exist infinitely many prime ideals  $P' \in \text{Spec}(R)$  such that  $P \subseteq P' \subseteq Q$  and  $\text{ht}(P'/P) = 1$ .*
- (b) *If  $\mathcal{M} \subseteq \text{Spec}(R)$  is an infinite set of prime ideals such that every  $P' \in \mathcal{M}$  satisfies  $P \subseteq P'$  and  $\text{ht}(P'/P) = 1$ , then  $P = \bigcap_{P' \in \mathcal{M}} P'$ .*

*Proof.* (a) By factoring out  $P$  and localizing at  $Q$  we may assume that  $R$  is a local domain with maximal ideal  $Q$ , and  $P = \{0\}$ . For  $a \in Q \setminus \{0\}$  there exists  $P' \in \text{Spec}(R)$  which is minimal over  $(a)_R$ . By the principal ideal theorem,  $P'$  has height one. So

$$Q \subseteq \bigcup_{\substack{P' \in \text{Spec}(R), \\ \text{ht}(P')=1}} P'.$$

If there existed only finitely many  $P' \in \text{Spec}(R)$  of height one, it would follow by the prime avoidance lemma (see [8, Lemma 7.7]) that  $Q$  is contained in one of them, contradicting the hypothesis  $\text{ht}(Q) > 1$

- (b) Clearly  $P \subseteq \bigcap_{P' \in \mathcal{M}} P' =: I$ . If  $P \subsetneq I$ , then every  $P' \in \mathcal{M}$  would be a minimal prime ideal over  $I$  and so  $\mathcal{M}$  would be finite (see [8, Corollary 3.14(d)]).  $\square$

**Lemma 2.3.** *A Noetherian ring  $R$  is a Jacobson ring if and only if for every  $P \in \text{Spec}(R)$  with  $\dim(R/P) = 1$  there exist infinitely many maximal ideals  $\mathfrak{m} \in \text{Spec}(R)$  with  $P \subseteq \mathfrak{m}$ .*

*Proof.* We prove that the negations of both statements are equivalent. First assume that  $R$  is not Jacobson. Choose  $P \in \text{Spec}(R)$  to be maximal among the prime ideals that are not intersections of maximal ideals. Then  $P$  itself is not maximal, so  $\dim(R/P) \geq 1$ . On the other hand, it is impossible that  $\dim(R/P) > 1$ , since by Lemma 2.2 that would imply that  $P$  is the intersection of strictly larger prime ideals and therefore (by its maximality) of maximal ideals. So  $\dim(R/P) = 1$ . Therefore every maximal ideal  $\mathfrak{m}$  containing  $P$  satisfies  $\text{ht}(\mathfrak{m}/P) = 1$ , so by Lemma 2.2(b) only finitely many such  $\mathfrak{m}$  exist.

Conversely, assume that  $R$  has a prime ideal  $P$  with  $\dim(R/P) = 1$  such that only finitely many maximal ideals, say  $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ , contain  $P$ . If  $P = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_n$ , then  $P$  would contain at least one of the  $\mathfrak{m}_i$ , so  $P$  would be maximal. Since this is not the case,  $R$  is not Jacobson.  $\square$

The following lemma may be surprising since it does not require the ring  $R$  to be Jacobson.

**Lemma 2.4.** *Let  $a$  be an element of a Noetherian ring  $R$  and set*

$$U_a := \{a^n(1 + ax) \mid n \in \mathbb{N}_0, x \in R\}$$

*Then the localization  $U_a^{-1}R$  is a Jacobson ring.*

*Proof.* We will use the criterion from Lemma 2.3 and the inclusion-preserving bijection between the prime ideals in  $S := U_a^{-1}R$  and the prime ideals  $P \in \text{Spec}(R)$  satisfying  $U_a \cap P = \emptyset$  (see [8, Theorem 6.5]). Let  $P \in \text{Spec}(R)$  with  $U_a \cap P = \emptyset$  such that  $\dim(S/U_a^{-1}P) = 1$ . Then there exists  $P_1 \in \text{Spec}(R)$  with  $P \subsetneq P_1$  and  $U_a \cap P_1 = \emptyset$ . The latter condition implies  $a \notin P_1$  and  $1 \notin P_1 + (a)_R$ , so there exists  $Q \in \text{Spec}(R)$  such that  $P_1 + (a)_R \subseteq Q$ . It follows that  $\text{ht}(Q/P) > 1$ , so by Lemma 2.2(a), the set

$$\mathcal{M} := \{P' \in \text{Spec}(R) \mid P \subseteq P' \subseteq Q, \text{ht}(P'/P) = 1\}$$

is infinite. Assume that the subset  $\mathcal{M}' := \{P' \in \mathcal{M} \mid a \in P'\}$  is also infinite. Then Lemma 2.2(b) would imply  $P = \bigcap_{P' \in \mathcal{M}'} P'$ , so  $a \in P$ , contradicting  $U_a \cap P = \emptyset$ . We conclude that  $\mathcal{M} \setminus \mathcal{M}'$  is infinite. Let  $P' \in \mathcal{M} \setminus \mathcal{M}'$ . Then  $P' \subseteq Q$  and  $a \notin P'$  imply that  $1 + ax \notin P'$  for every  $x \in R$ , so  $U_a \cap P' = \emptyset$ . Therefore  $U_a^{-1}P' \in \text{Spec}(S)$ , and we also have  $U_a^{-1}P \subsetneq U_a^{-1}P'$ . Since  $\dim(S/U_a^{-1}P) = 1$ , this implies that  $U_a^{-1}P'$  is a maximal ideal. So by Lemma 2.3, the infinity of  $\mathcal{M} \setminus \mathcal{M}'$  implies that  $S$  is a Jacobson ring.  $\square$

**Remark.** The localization  $U_a^{-1}R$  from Lemma 2.4 was also used by Coquand and Lombardi [3]. They called it the *boundary* of  $a$  in  $R$ .  $\triangleleft$

**Lemma 2.5.** *Let  $R$  be a Jacobson ring and let  $A$  be a finitely generated  $R$ -algebra that is a field. Then the kernel of the map  $R \rightarrow A$  is a maximal ideal.*

*Proof.* We may assume that the map  $R \rightarrow A$  is injective, so we may view  $R$  as a subring of  $A$ . We need to show that  $R$  is a field. Since  $A$  is finitely generated as an algebra over  $K := \text{Quot}(R)$  (the field of fractions), it is algebraic over  $K$  (see [8, Lemma 1.1(b)]). So  $A$  is a finite field extension of  $K$ , and choosing a basis yields a map

$$\varphi: A \rightarrow K^{n \times n}$$

sending each  $a \in A$  to the representation matrix of the linear map given by multiplication by  $a$ . Let  $a_1, \dots, a_n$  be generators of  $A$  as an  $R$ -algebra and choose  $b \in R \setminus \{0\}$  to be a common denominator of all matrix entries of all  $\varphi(a_i)$ . Then  $\varphi(a_i) \in R[b^{-1}]^{n \times n}$ , and since  $\varphi$  is a homomorphism of  $R$ -algebras, its image is contained in  $R[b^{-1}]^{n \times n}$ . If  $a \in R \setminus \{0\}$ , then  $a^{-1} \in A$ , so

$$\text{diag}(a^{-1}, \dots, a^{-1}) = \varphi(a^{-1}) \in R[b^{-1}]^{n \times n}.$$

This implies  $a^{-1} \in R[b^{-1}]$ , so  $R[b^{-1}]$  is a field. From this it follows by Eisenbud [4, Lemma 4.20] that  $R$  is a field.  $\square$

As announced, we can now prove the upper bound for the transcendence degree. As above,  $\text{lex}$  stands for the lexicographic ordering with  $x_i > x_{i+1}$  for all  $i$ .

**Proposition 2.6.** (a) *If  $R$  is a ring, then*

$$\text{trdeg}_{\text{lex}}(R) \leq \dim(R).$$

(b) *If  $A$  is a finitely generated algebra over a Noetherian Jacobson ring  $R$ , then*

$$\text{trdeg}_{\text{lex}}(A : R) \leq \dim(A).$$

**Remark.** Part (a) is contained in Coquand and Lombardi's result (Theorem 1.3). We include a proof of this part for the reader's convenience.  $\triangleleft$

*Proof of Proposition 2.6.* We prove both parts simultaneously, setting  $A = R$  in the case of part (a). It is clear that without loss of generality we may assume the map  $R \rightarrow A$  to be injective, so we may view  $R$  as a subring of  $A$ . We may also assume  $A \neq \{0\}$  and  $\dim(A) < \infty$ . We use induction on  $n := \dim(A) + 1$ .

Let  $a_1, \dots, a_n \in A$ . Consider the multiplicative set

$$U := \{f(a_n) \mid f \in R[x] \text{ is submonic}\} \subseteq A$$

and set  $A' := U^{-1}A$ . By way of contradiction, assume  $\dim(A') \geq n - 1$ . Using the correspondence between prime ideals in  $A'$  and prime ideals in  $A$  that do not intersect with  $U$ , we obtain a chain  $P_1 \subsetneq P_2 \subsetneq \dots \subsetneq P_n$  with  $P_i \in \text{Spec}(A)$  and  $U \cap P_i = \emptyset$ . Since  $\dim(A) = n - 1$ ,  $A/P_n$  must be a field. In the case of part (b), it follows by Lemma 2.5 that  $R \cap P_n \subseteq R$  is a maximal ideal. In the case of part (a), this is also true since  $R = A$ . So  $A/P_n$  is an algebraic field extension of  $R/(R \cap P_n)$  (see [8, Lemma 1.1(b)]). From  $U \cap P_n = \emptyset$  we conclude that  $a_n + P_n \in A/P_n$  is invertible. So there exists  $g \in R[x]$  such that  $a_n g(a_n) - 1 \in P_n$ . But  $1 - xg$  is submonic, so  $1 - a_n g(a_n) \in U$ , contradicting  $U \cap P_n = \emptyset$ .

We conclude that  $\dim(A') + 1 < n - 1$ . If  $A' = \{0\}$  (which must happen if  $n = 1$ ), then  $0 \in U$ , so  $a_n$  satisfies a submonic equation and therefore  $a_1, \dots, a_n$  are algebraically dependent with respect to  $\text{lex}$ . Having dealt with this case, we may assume  $A' \neq \{0\}$ .

Clearly  $R' := U^{-1}R[a_n] \subseteq A'$ . In the case of part (a) we have  $R' = A'$ . In the case of part (b),  $A'$  is finitely generated as an  $R'$ -algebra, and by Lemma 2.4,  $R'$  is a Jacobson ring. So in both cases the induction hypothesis tells us that  $\frac{a_1}{1}, \dots, \frac{a_{n-1}}{1} \in A'$  satisfy a polynomial  $\tilde{f} \in R'[x_1, \dots, x_{n-1}]$  that is submonic with respect to  $\text{lex}$ . Multiplying the coefficients of  $\tilde{f}$  by a suitable element from  $U$ , we obtain  $\hat{f} \in R[a_n][x_1, \dots, x_{n-1}]$  whose trailing coefficient  $c_t \in R[a_n]$  lies in  $U$  such that  $\hat{f}(a_1, \dots, a_{n-1}) = 0$ . By replacing every coefficient  $c \in R[a_n]$  of  $\hat{f}$  by a  $c' \in R[x_n]$  with  $c'(a_n) = c$ , we obtain  $f \in R[x_1, \dots, x_n]$  with  $f(a_1, \dots, a_n) = 0$ . Since  $c_t \in U$ , we may choose the coefficient  $c'_t \in R[x_n]$  of  $f$  to be submonic. The trailing coefficient of  $f$  (with respect to  $\text{lex}$ ) is equal to the trailing coefficient of  $c'_t$ , which is 1. Therefore  $f$  is submonic with respect to  $\text{lex}$ .

We conclude that  $a_1, \dots, a_n$  are algebraically dependent with respect to  $\text{lex}$ . Since they were chosen as arbitrary elements of  $A$ , this shows that  $\text{trdeg}_{\text{lex}}(A : R) \leq n - 1 = \dim(A)$ .  $\square$

We can now finish the proof of Theorem 1.4.

*Proof of Theorem 1.4(b) and (c).* Theorem 1.3 and Definition 1.1 imply

$$\dim(A) = \text{trdeg}_{\text{lex}}(A) \leq \text{trdeg}_{\text{lex}}(A : R).$$

If  $A$  is Noetherian, (2.1) and (2.4) yield the finer inequality

$$\dim(A) \leq \text{trdeg}(A) \leq \text{trdeg}(A : R) \leq \text{trdeg}_{\text{lex}}(A : R).$$

So for the proof of (b) and (c) it suffices to show that  $\text{trdeg}_{\text{lex}}(A : R) \leq \dim(A)$ . We may assume that  $0 \leq \dim(A) < \infty$  and write  $n = \dim(A) + 1$ .

By hypothesis,  $A$  is a subalgebra of a finitely generated  $R$ -algebra  $B$ . Let  $P_1, \dots, P_r \in \text{Spec}(B)$  be the minimal prime ideals of  $B$ , and assume that we can show  $\text{trdeg}_{\text{lex}}(A/A \cap P_i : R) \leq \dim(A/A \cap P_i)$  for all  $i$ . Then for  $a_1, \dots, a_n \in A$  there exist polynomials  $f_1, \dots, f_r \in R[x_1, \dots, x_n]$  that are submonic with respect to lex such that  $f_i(a_1, \dots, a_n) \in P_i$ . So  $\prod_{i=1}^r f_i(a_1, \dots, a_n)$  lies in the nilradical of  $B$ , hence there exists  $k$  such that  $a_1, \dots, a_n$  satisfy the polynomial  $\prod_{i=1}^r f_i^k$ , which is also submonic with respect to lex. This shows that we may assume  $B$  to be an integral domain.

By Giral [5, Proposition 2.1(b)] (or [8, Exercise 10.3]), there exists a nonzero  $a \in A$  such that  $A[a^{-1}]$  is finitely generated as an  $R$ -algebra. So we may apply Proposition 2.6(b) and get  $\text{trdeg}_{\text{lex}}(A[a^{-1}] : R) \leq \dim(A[a^{-1}])$ . We obtain

$$\text{trdeg}_{\text{lex}}(A : R) \leq \text{trdeg}_{\text{lex}}(A[a^{-1}] : R) \leq \dim(A[a^{-1}]) \leq \dim(A),$$

where the first inequality follows directly from Definition 1.1, and the last since  $A[a^{-1}]$  is a localization of  $A$ . This completes the proof.  $\square$

**Remark 2.7.** (a) The hypothesis that  $R$  be a Jacobson ring cannot be dropped from Theorem 1.4(b) and (c). In fact, the validity of Theorem 1.4(b) and (c) characterizes Jacobson rings in the following sense: If  $R$  is a non-Jacobson ring, then by Eisenbud [4, Lemma 4.20],  $R$  has a nonmaximal prime ideal  $P$  such  $S := R/P$  contains a nonzero element  $b$  for which  $A := S[b^{-1}]$  is a field. So  $\dim(A) = 0$ , but  $b$  is not a unit in  $S$ , so Example 1.2(1) yields

$$1 \leq \text{trdeg}(S) = \text{trdeg}(S : R) \leq \text{trdeg}(A : R) \leq \text{trdeg}_{\text{lex}}(A : R).$$

Since  $A$  is a finitely generated  $R$ -algebra, the assertions of Theorem 1.4(b) and (c) fail for  $R$ .

(b) Neither can the hypothesis that  $A$  is subfinite be dropped. In fact, if  $R$  is any nonzero ring, we can choose a maximal ideal  $\mathfrak{m}$  of  $R$  and form the polynomial ring  $S := (R/\mathfrak{m})[x]$  and  $A := \text{Quot}(S)$ . Then

$$\dim(A) = 0 < 1 = \text{trdeg}(A : R/\mathfrak{m}) = \text{trdeg}(A : R) = \text{trdeg}_{\text{lex}}(A : R).$$

### 3 Some applications and further results

It seems to be rare that Theorem 1.4 helps to compute the dimension of rings. In fact, the transcendence degree seems to be the less accessible quantity in most cases, so knowledge of the dimension provides information about the structure of the ring that is encoded in the transcendence degree. Example 1.2(4) contains such an instance. Here is a further example.

*Example 3.1.* Let  $a$  and  $b$  be two nonzero algebraic numbers (i.e., elements of an algebraic closure of  $\mathbb{Q}$ ). There exists  $d \in \mathbb{Z} \setminus \{0\}$  such that  $a$  and  $b$  are integral over  $\mathbb{Z}[d^{-1}]$ , so  $A := \mathbb{Z}[a, b, d^{-1}]$  has Krull dimension 1. By Theorem 1.4(b),  $\text{trdeg}_{\text{lex}}(A : \mathbb{Z}) = 1$ , so  $a, b$  satisfy a polynomial  $f \in \mathbb{Z}[x_1, x_2]$  that is submonic with respect to lex. If  $x_1^m x_2^n$  is the trailing monomial of  $f$ , then all monomials of  $f$  are divisible by  $x_1^m$ , so we may assume  $m = 0$ . We obtain

$$b^n = a \cdot g(a, b) + b^{n+1} \cdot h(a, b)$$

with  $g, h \in \mathbb{Z}[x_1, x_2]$  polynomials. It is not so clear how the existence of such a relation follows directly from the properties of algebraic numbers.  $\triangleleft$

The following corollary of Theorem 1.4(b) may be new:

**Corollary 3.2.** *Let  $R$  be a Noetherian Jacobson ring,  $B$  a subfinite  $R$ -algebra, and  $A \subseteq B$  a subalgebra. Then*

$$\dim(A) \leq \dim(B).$$



*Example 3.3.* Let  $R$  be a ring that is finitely generated as a  $\mathbb{Z}$ -algebra,  $G \subseteq \text{Aut}(R)$  a group of automorphisms of  $R$  and  $H \subseteq G$  a subgroup. Then Corollary 3.2 tells us that

$$\dim(R^G) \leq \dim(R^H),$$

even though the invariant rings need not be finitely generated (see Nagata [10]).  $\triangleleft$

We also get the following geometric-topological consequence.

**Theorem 3.4.** *Let  $X$  be a scheme of finite type over a Noetherian Jacobson ring  $R$ . Let  $Y$  be a locally closed subset of the underlying topological space of  $X$ . Then  $\dim(Y) = \dim(\overline{Y})$ .*

*Proof.* By part (b) of the following Lemma 3.5, we need to show that if  $\dim(\overline{Y}) \geq n$  for an integer  $n$ , then  $\dim(Y) \geq n$ . (We will invoke Lemma 3.5(b) numerous times during this proof without always mentioning it.) By Lemma 3.5(d), there exists an open affine subset  $U$  of  $X$  such that  $\dim(U \cap \overline{Y}) \geq n$ . By Lemma 3.5(c),  $U \cap \overline{Y}$  is the closure of  $U \cap Y$  in  $U$ . It is also clear that  $U \cap Y$  is locally closed in  $U$ . Moreover, if  $U = \text{Spec}(A)$ , then  $A$  is finitely generated as an  $R$ -algebra (see Hartshorne [7, Chapter II, Exercise 3.3(c)]). So by substituting  $X$  by  $U$  and  $Y$  by  $U \cap Y$ , we may assume that  $X = \text{Spec}(A)$  with  $A$  a finitely generated  $R$ -algebra.

Since  $\overline{Y} = \text{Spec}(B)$  with  $B$  a quotient ring of  $A$ , we may substitute  $A$  by  $B$  and assume that  $Y$  is dense in  $X$ .  $X$  has an irreducible component  $X_i$  with  $\dim(X_i) \geq n$ . Since  $X_i \cap Y$  is nonempty,  $X_i \cap Y$  is dense in  $X_i$ . So by substituting  $X$  by  $X_i$  and factoring out the nilradical of  $A$ , we may assume  $A$  to be an integral domain. Since  $Y \subseteq X$  is nonempty and open, there exists a nonzero ideal  $I \subseteq A$  such that

$$Y = \{P \in \text{Spec}(A) \mid I \not\subseteq P\}.$$

Choose  $0 \neq a \in I$ . Then

$$D_a := \{P \in \text{Spec}(A) \mid a \notin P\} \subseteq Y,$$

so it suffices to show that  $\dim(D_a) \geq n$ . But  $D_a$  is homeomorphic to  $\text{Spec}(A[a^{-1}])$  (see [8, Theorem 6.5 and Exercise 6.5]). Since  $A$  is a Jacobson ring (see Eisenbud [4, Theorem 4.19]), Corollary 3.2 yields

$$\dim(A[a^{-1}]) \geq \dim(A),$$

so  $\dim(Y) \geq \dim(D_a) = \dim(A[a^{-1}]) \geq n$ .  $\square$

The following lemma was used in the previous proof.

**Lemma 3.5.** *Let  $X$  be a topological space and  $Y \subseteq X$  a subset equipped with the subspace topology.*

- (a)  *$Y$  is irreducible if and only if its closure  $\overline{Y}$  is irreducible.*
- (b) *The inequality  $\dim(Y) \leq \dim(X)$  holds for the dimensions of  $X$  and  $Y$  as topological spaces.*
- (c) *If  $U \subseteq X$  is an open subset, then  $U \cap \overline{Y}$  is the closure of  $U \cap Y$  in  $U$ .*
- (d) *If  $\mathcal{A}$  is a set of open subsets of  $X$  such that  $X = \bigcup_{U \in \mathcal{A}} U$  and if  $\dim(Y) \geq n$  holds for an integer  $n$ , then there exists  $U \in \mathcal{A}$  such that  $\dim(U \cap Y) \geq n$ .*

*Proof.* The proofs of (a) and (b) are straightforward and left to the reader.

To prove (c), let  $Z \subseteq X$  be closed with  $U \cap Y \subseteq Z$ . Then  $\overline{Y} \subseteq (\overline{Y} \setminus U) \cup Z$ , so  $U \cap \overline{Y} \subseteq U \cap Z$ . This shows that  $U \cap \overline{Y}$  is the smallest subset of  $X$  that contains  $U \cap Y$  and is closed in  $U$ .

Under the hypothesis of (d), we may assume  $Y = X$  since  $\{Y \cap U \mid U \in \mathcal{A}\}$  is an open covering of  $Y$ . There exists a chain  $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n$  of closed irreducible subsets of  $X$ . We can choose  $U \in \mathcal{A}$  with  $U \cap X_0 \neq \emptyset$ . Then  $U_i := U \cap X_i \neq \emptyset$ , and the  $U_i$  form an ascending chain of closed subsets in  $U$ . Since  $U_i$  is nonempty and open in  $X_i$ , its closure equals  $X_i$ . This implies that the inclusions between the  $U_i$  are strict, and, by (a), that the  $U_i$  are irreducible. Therefore  $\dim(U) \geq n$ .  $\square$

**Remark.** Jacobson rings are characterized by the validity of Theorem 3.4. Indeed, every non-Jacobson ring  $R$  has a nonmaximal prime ideal  $P$  such  $S := R/P$  contains a nonzero element  $b$  for which  $S[b^{-1}]$  is a field (see Remark 2.7(a)). Then with  $X := \text{Spec}(S)$  und  $Y := \{Q \in X \mid b \notin Q\}$  we have  $\dim(Y) = 0$ , but  $\overline{Y} = X$  and  $\dim(X) > 0$ .  $\triangleleft$

It seems odd that the lexicographic ordering plays such a special role in Theorems 1.3 and 1.4. Can it be substituted by other monomial orderings? Theorem 3.7 below gives answers in some special cases. We need some preparations for its proof.

Let us call a monomial ordering “ $\preceq$ ” **weight-graded** if there exist positive real numbers  $w_1, w_2, \dots$  such that if  $\prod_{i=1}^n x_i^{d_i} \preceq \prod_{i=1}^n x_i^{e_i}$  (with  $n, d_i, e_i \in \mathbb{N}_0$ ), then  $\sum_{i=1}^n w_i d_i \leq \sum_{i=1}^n w_i e_i$ .

**Lemma 3.6.** *Let  $a_1, \dots, a_n$  be elements of a Noetherian ring  $R$  such that  $\dim(R) < n$  and  $\dim(R/(a_1, \dots, a_n)_R) \leq 0$ . Then  $a_1, \dots, a_n$  are algebraically dependent with respect to every weight-graded monomial ordering.*

*Proof.* Let “ $\preceq$ ” be a weight-graded monomial ordering. Consider the set  $J \subseteq R$  of all trailing monomials of polynomials  $f \in R[x_1, \dots, x_n]$  with  $f(a_1, \dots, a_n) = 0$ . It is easy to see that  $J$  is an ideal and  $I := (a_1, \dots, a_n)_R \subseteq J$ . We need to show that  $J = R$ . Suppose that we can show that for all maximal ideals  $\mathfrak{m} \in \text{Spec}(R)$  with  $I \subseteq \mathfrak{m}$ , the elements  $\frac{a_i}{1} \in R_{\mathfrak{m}}$  are algebraically dependent with respect to “ $\preceq$ ”. Then  $J \cap (R \setminus \mathfrak{m})$  is nonempty, so  $J = R$ . This shows that we can assume that  $R$  is a local ring and  $I$  is contained in its maximal ideal.

By hypothesis,  $R/I$  is Artinian, so by Matsumura [9, Theorem 17], the length of  $R/I^j$  is given by a polynomial of degree  $\dim(R) < n$  for  $j$  large enough. Since multiplying all weights  $w_i$  by the same positive constant does not change the monomial ordering, we may assume that  $w_i \geq 1$  for  $i \in \{1, \dots, n\}$ . For  $(\underline{e}) = (e_1, \dots, e_n) \in \mathbb{N}_0^n$  write  $w(\underline{e}) := \sum_{i=1}^n w_i e_i$ , and for  $j \in \mathbb{N}_0$  set

$$I_j := \left( \prod_{i=1}^n a_i^{e_i} \mid (\underline{e}) \in \mathbb{N}_0^n, w(\underline{e}) \geq j \right)_R.$$

Then  $I^j \subseteq I_j$ , so the length of  $R/I_j$  is bounded above by a polynomial of degree  $< n$ . Take  $j \in \mathbb{N}_0$  and consider the set

$$A_j := \{(\underline{d}) \in \mathbb{N}_0^n \mid j \leq w(\underline{d}) < j+1\}.$$

By way of contradiction, assume that the  $a_i$  are algebraically independent with respect to “ $\preceq$ ”. For  $(\underline{d}) \in A_j$ , this assumption and the fact that “ $\preceq$ ” is weight-graded imply

$$I_{j+1} \subseteq \left( \prod_{i=1}^n a_i^{e_i} \mid \prod_{i=1}^n x_i^{e_i} \succ \prod_{i=1}^n x_i^{d_i} \right)_R \subsetneq \left( \prod_{i=1}^n a_i^{e_i} \mid \prod_{i=1}^n x_i^{e_i} \succeq \prod_{i=1}^n x_i^{d_i} \right)_R \subseteq I_j.$$

So ordering  $A_j$  according to “ $\preceq$ ” yields a chain of length  $|A_j|$  of ideals between  $I_{j+1}$  and  $I_j$ . This implies  $\text{length}(I_j/I_{j+1}) \geq |A_j|$ , so  $\text{length}(R/I_j) \geq |\{(\underline{d}) \in \mathbb{N}_0^n \mid w(\underline{d}) < j\}|$ . With  $\overline{w} := \max\{w_1, \dots, w_n\}$ , we obtain

$$\text{length}(R/I_j) \geq |\{(\underline{d}) \in \mathbb{N}_0^n \mid \overline{w}(d_1 + \dots + d_n) < j\}| = \binom{\lceil j/\overline{w} \rceil - 1 + n}{n},$$

contradicting the fact that the length of  $R/I_j$  is bounded above by a polynomial of degree  $< n$ . This contradiction finishes the proof.  $\square$

**Theorem 3.7.** *Let  $A$  be a Noetherian ring and let “ $\preceq$ ” be a monomial ordering.*

(a) *If  $A$  is an algebra over a ring  $R$  that contains a field  $K$  such that  $A$  is a subfinite  $K$ -algebra, then*

$$\text{trdeg}_{\preceq}(A : R) = \dim(A).$$

(b) *If  $A = B_P$  with  $B$  a finitely generated algebra over a field and  $P \in \text{Spec}(B)$ , then*

$$\text{trdeg}_{\preceq}(A) = \dim(A).$$

(c) If  $\dim(A) \leq 1$ , then

$$\text{trdeg}_{\preceq}(A) = \dim(A).$$

(d) If  $\dim(A) \leq 1$  and  $A$  is an subfinite algebra over a Noetherian Jacobson ring  $R$ , then

$$\text{trdeg}_{\preceq}(A : R) = \dim(A).$$

*Proof.* By Definition 1.1 we may assume  $A \neq \{0\}$ . By (2.1) and Definition 1.1 we have

$$\dim(A) \leq \text{trdeg}(A) \leq \text{trdeg}_{\preceq}(A) \leq \text{trdeg}_{\preceq}(A : R),$$

where we set  $R = A$  in the case of (b) and (c). So we may assume  $n := \dim(A) + 1 < \infty$  and need to show that all  $a_1, \dots, a_n \in A$  are algebraically dependent over  $R$  with respect to “ $\preceq$ ”.

- (a) By Theorem 1.4(c), the  $a_i$  are algebraically dependent over  $K$ . Since  $K$  is a field, the algebraic dependence is with respect to “ $\preceq$ ”, and since  $K \subseteq R$ , it is over  $R$ .
- (b) Let  $P_1, \dots, P_r \in \text{Spec}(A)$  be the minimal prime ideals in  $A$ . For each  $i$ ,  $A/P_i$  is the localization of a finitely generated domain over a field at a prime ideal, so by Tanimoto [13],  $A/P_i$  contains a field  $K_i$  such that  $\text{trdeg}(A/P_i : K_i) = \dim(A/P_i)$ . Therefore

$$\text{trdeg}_{\preceq}(A/P_i) \leq \text{trdeg}_{\preceq}(A/P_i : K_i) = \text{trdeg}(A/P_i : K_i) = \dim(A/P_i) \leq \dim(A).$$

So we obtain polynomials  $f_i \in A[x_1, \dots, x_n]$  that are submonic with respect to “ $\preceq$ ” such that  $f_i(a_1, \dots, a_n) \in P_i$ . A power of the product of the  $f_i$  yields a submonic equation for  $a_1, \dots, a_n$ .

- (c) If  $\dim(A) = 0$ , the algebraic dependence of the  $a_i$  follows from Theorem 1.3 since every monomial ordering restricts to the lexicographic ordering on  $A[x_1]$ . So we may assume  $\dim(A) = 1$ . It follows by Robbiano [12] that the restriction of “ $\preceq$ ” to  $A[x_1, x_2]$  is either a lexicographic ordering or weight-graded. By Theorem 1.3 we may assume the latter.

Let  $P_1, \dots, P_r$  be the minimal prime ideals of  $A$  satisfying  $a_1 \notin P_i$  and set  $I := \bigcap_{i=1}^r P_i$  (with  $I := R$  in the case  $r = 0$ ). Every  $P \in \text{Spec}(A)$  with  $I \subseteq P$  contains at least one of the  $P_i$ , and if  $a_1 \in P$ , then  $P_i \not\subseteq P$ , so  $\dim(A/P) = 0$ . This shows that  $\dim(A/(I + Aa_1)) \leq 0$ . Applying Lemma 3.6 to  $R := A/I$  yields  $f \in A[x_1, x_2]$  that is submonic with respect to “ $\preceq$ ” such that  $f(a_1, a_2) \in I$ . By multiplying  $f$  by  $x_1$ , we may assume  $f(a_1, a_2) \in Aa_1$ , so by the definition of  $I$ ,  $f(a_1, a_2)$  lies in every minimal prime ideal of  $A$ . Therefore a suitable power of  $f$  yields a submonic equation for  $a_1, a_2$ .

- (d) By Corollary 3.2, the subalgebra  $A' \subseteq A$  generated by the  $a_i$  has dimension at most 1. By (c), the  $a_i$  satisfy an equation  $f \in A'[x_1, \dots, x_n]$  that is submonic with respect to “ $\preceq$ ”. Obtain  $\hat{f} \in R[x_1, \dots, x_n]$  from  $f$  by replacing every coefficient  $c \in A'$  of  $f$  by a  $\hat{c} \in R[x_1, \dots, x_n]$  with  $\hat{c}(a_1, \dots, a_n) = c$ , where the trailing coefficient of  $f$  is replaced by  $1 \in R$ . It follows that  $\hat{f}$  is submonic with respect to “ $\preceq$ ” and  $\hat{f}(a_1, \dots, a_n) = 0$ .  $\square$

In view of Theorem 3.7, a candidate that comes to mind for a ring  $R$  such that  $\text{trdeg}_{\preceq}(R) > \dim(R)$  for some monomial ordering “ $\preceq$ ” is the polynomial ring  $\mathbb{Z}[x]$ . Using a short program written in MAGMA [2], I tested millions of randomly selected triples of polynomials from  $\mathbb{Z}[x]$  and verified that they were all algebraically dependent with respect to the graded reverse lexicographic ordering, even over the subring  $\mathbb{Z}$ . This prompts the following conjecture:

**Conjecture 3.8.** *Theorem 1.4(a) and (c) holds with “trdeg” replaced by “trdeg $_{\preceq}$ ”, with “ $\preceq$ ” an arbitrary monomial ordering.*

So far, all efforts to prove the conjecture have been futile. Let me mention that it would follow if one could get rid of the hypothesis “ $\dim(R/(a_1, \dots, a_n)_R) \leq 0$ ” in Lemma 3.6.

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